On the Choice of Parametric Families of Copulas

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Vienna, July 2008

Outline

1. Brief Review of Copulas
   - What is a Copula and Why should we care?
2. Copula misspecification
   - Simulation study of the effects of copula misspecification
3. Choice of a Copula Family
   - A nonparametric estimate of distributional distances

Copulas

- Copulas present one possible approach to model dependence.
- If X, Y are continuous random variables with distribution functions (df) F_X and, respectively, F_Y we specify the joint df using the copula $C : [0,1] \times [0,1] \rightarrow [0,1]$ such that

  $$F_{XY}(F_X^{-1}(u), F_Y^{-1}(v)) = \Pr(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(v)) = C(u,v).$$

- The copula C bridges the marginal distributions of X and Y. Interesting: connection between dependence structures and various families of copulas.
- Popular class: *Archimedean copulas*

  $$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)),$$

  where ϕ is a continuous, strictly decreasing function $\phi : [0,1] \rightarrow [0.\infty]$ and

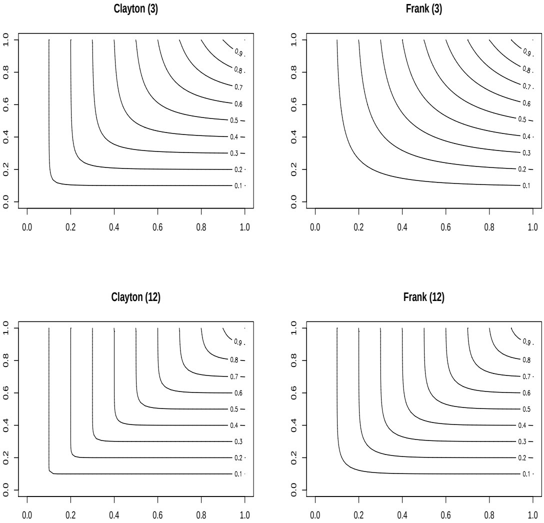
  $$\phi^{[-1]} = \begin{cases} \phi^{-1}(t) & \textit{if} \;\; 0 \leq t \leq \phi(0) \\ \phi(0) & \textit{if} \;\; \phi(0) \leq t \leq \infty. \end{cases}$$

Copulas (cont'd)

- Examples:
  Clayton's copula: $C(u,v) = \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}$.
  Frank's copula: $C(u,v) = -\frac{1}{\theta}\ln\left[1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right]$.
- For the purpose of inference, given a family of copulas has been selected, of interest is the estimation of θ as well as the marginal distributions' parameters, say λ_X, λ_Y.
- The effect of marginal models misspecification has been well documented. Also important is the effect of copula misspecification, especially when of interest are conditional estimates such as $\mathrm{E}[X|Y=y]$, $\mathrm{Var}(X|Y=y)$.
- Central to the performance of the model is the correct specification of the copula family.

Copulas (cont’d)

Contour plots of the bivariate cdf:



Copula Misspecification: A simulation study

- We assume that the marginals are known.
- We generate data following the bivariate Clayton's density.
- We fit a model using Frank's copula. We are interested in evaluating the bias for conditional mean and variance estimators.
- Each simulation study has a sample size of $n = 500$ and we replicate each study $K = 200$ times.
- The conditional means are computed via Monte Carlo using a sample of size $M = 5000$.

Simulation Results

<table>
<tr><td colspan="5">Clayton's $\theta = 3$; $F_X = \text{Exp}(2)$, $F_Y = \text{Exp}(1)$</td></tr>
<tr><td>y_0</td><td>0.5</td><td>1.0</td><td>1.5</td><td>2.5</td></tr>
<tr><td>$B(\mu_{y_0})$</td><td>-0.067 (0.009)</td><td>-0.072 (0.014)</td><td>-0.003 (0.022)</td><td>0.140 (0.037)</td></tr>
<tr><td>$B(\sigma^2_{y_0})$</td><td>0.142 (0.026)</td><td>0.364 (0.043)</td><td>0.646 (0.080)</td><td>1.041 (0.147)</td></tr>
<tr><td colspan="5">Clayton's $\theta = 3$; $F_X = F_Y = \text{Weibull}(1, 2)$</td></tr>
<tr><td>y_0</td><td>0.5</td><td>1.0</td><td>1.5</td><td>2.5</td></tr>
<tr><td>$B(\mu_{y_0})$</td><td>-0.052 (0.042)</td><td>-0.285 (0.048)</td><td>-0.357 (0.051)</td><td>-0.170 (0.071)</td></tr>
<tr><td>$B(\sigma^2_{y_0})$</td><td>-0.061(0.018)</td><td>-0.647 (0.209)</td><td>-1.036 (0.279)</td><td>-1.030 (0.400)</td></tr>
<tr><td colspan="5">Clayton's $\theta = 12$; $F_X = F_Y = \text{Weibull}(1, 2)$</td></tr>
<tr><td>y_0</td><td>0.5</td><td>1.0</td><td>1.5</td><td>2.5</td></tr>
<tr><td>$B(\mu_{y_0})$</td><td>0.011 (0.012)</td><td>-0.008(0.016)</td><td>-0.035 (0.023)</td><td>-0.118 (0.047)</td></tr>
<tr><td>$B(\sigma^2_{y_0})$</td><td>0.056 (0.006)</td><td>0.076 (0.014)</td><td>0.050 (0.043)</td><td>-0.294 (0.095)</td></tr>
</table>

Outline of the approach proposed

- Problem: Given a sample $\{x_i, y_i\}_{1\leq i\leq n}$ choose the family of copulas that best approximates the true unknown joint density $c^*(x, y)$.
- Assume marginals are known and (without loss of generality) Uniform$(0, 1)$.
- Compute a nonparametric estimate of the two-dimensional density.
- Among a set of possible families find the one who is closest (wrt a certain distributional distance) to the nonparametric estimate.
- Compare two different discrepancies: Kullback-Leibler and Hellinger.

Nonparametric Estimate

- A sample of size n from c^*: $\{(u_i, v_i) \in [0,1]^2 : 1 \leq i \leq n\}$.
- The kernel density is defined by $\hat{c}^*(x; H) = n^{-1} \sum_{i=1}^{n} K_H(x - X_i)$, where $x = (x_1, x_2)^T$, $X_i = (u_i, v_i)$ and $K_H(x) = |H|^{-1/2} K(H^{-1/2} x)$.
- H is non-diagonal since there is a large probability mass oriented away from the coordinate directions
- H is data-driven (least squares cross-validation).

Distributional Distances

- Kullback-Leibler discrepancy is defined as

$$KL(f,g) = \int \log(f(x)/g(x))f(x)dx.$$

- The Hellinger distance is

$$HE^2(f,g) = \int f(x)\left[1 - \frac{\sqrt{g(x)}}{\sqrt{f(x)}}\right]^2 dx.$$

Computing the distance

- Two families of copula densities $\mathcal{A}= \{c_\alpha : \alpha \in A\}$ and $\mathcal{B}= \{c_\beta : \beta \in B\}$, where α and β are copula parameters.
- Find the MLE's $\hat{\alpha}$ and $\hat{\beta}$.
- Generate a sample $\{(\tilde{u}_i, \tilde{v}_i) : 1 \leq i \leq m\}$ drawn from $c_{\hat{\alpha}}$
- Compute

$$\widehat{KL}(c_{\hat{\theta}}, \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m} c_{\hat{\theta}}(\tilde{u}_i, \tilde{v}_i)[\log(c_{\hat{\theta}}(\tilde{u}_i, \tilde{v}_i)) - \log(\hat{c}^*(\tilde{u}_i, \tilde{v}_i))],$$

  $\theta = \alpha, \beta.$

- Similarly for the Hellinger distance:

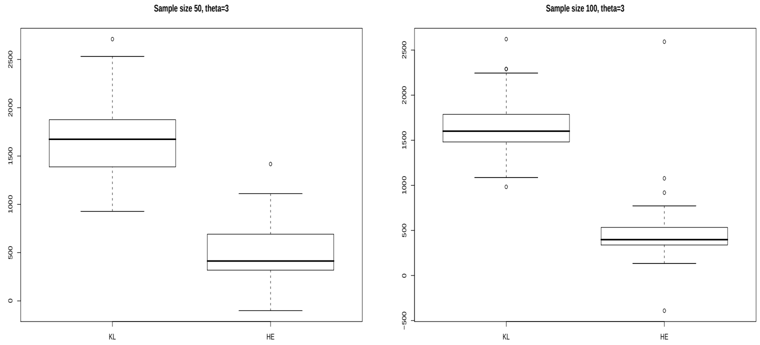
$$\widehat{HE^2}(c_{\hat{\theta}}, \hat{c}^*) = \frac{1}{m}\sum_{i=1}^{m}\left[1 - \frac{\sqrt{\hat{c}^*(\tilde{u}_i, \tilde{v}_i)}}{\sqrt{c_{\hat{\theta}}(\tilde{u}_i, \tilde{v}_i)}}\right]^2, \quad \theta = \alpha, \beta.$$

Simulation Results

Method\\n	50	100	300	500
Clayton's $\theta = 3$				
KL	100	100	100	100
HE	99	99	100	100
Clayton's $\theta = 8$				
KL	100	100	100	100
HE	100	100	100	100
Clayton's $\theta = 12$				
KL	100	100	100	100
HE	100	100	100	100

Further Comparison

Compare difference in distances measured by KL and HE ($\theta = 3$).



Further Comparison

Difference in distances measured by KL and HE ($\theta = 8, 12$).